Holomorphic Quantization on the Torus and Finite Quantum Mechanics

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Abstract

We construct explicitly the quantization of classical linear maps of $SL(2,\mathbb{R})$ on toroidal phase space, of arbitrary modulus, using the holomorphic (chiral) version of the metaplectic representation. We show that Finite Quantum Mechanics (FQM) on tori of arbitrary integer discretization, is a consistent restriction of the holomorphic quantization of $SL(2,\mathbb{Z})$ to the subgroup $SL(2,\mathbb{Z})/\Gamma_l$, Γ_l being the principal congruent subgroup mod l, on a finite dimensional Hilbert space. The generators of the "rotation group" mod l, $O_l(2) \subset SL(2,l)$, for arbitrary values of l are determined as well as their quantum mechanical eigenvalues and eigenstates.

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1 Introduction

A most fascinating branch of mathematics, Number Theory [1], quite unexpectedly, has made its appearance in a variety of research areas in physics the last fifteeen years. In classical and quantum chaos [2, 3, 4, 5], localization in incommensurate lattices [6], classification of rational conformal field theories [7, 8], in string theory [9], etc. (cf. also ref. [10]).

On the other hand, Number Theory, as is well known, is, for many years, an important tool in theoretical computer science (algorithms, cryptography) as well as in signal processing [11, 12].

Motivated by recent work, on the information paradox of black holes [13], as well as by indications that string theory predicts an absolute minimum distance in nature, of the order of $M_{\text{Planck}}^{-1} \approx 10^{-33}\,\text{cm}$ [14], two of the authors reconsidered the ancient question, why nature has to use real and complex numbers, once there is a fundamental unit of length, and they proposed to study quantum mechanics over finite sets of integers with the structure of finite algebraic fields, hoping to be able, eventually, to formulate field and string theory over them [15], which possess the property of containing finite information per unit of physical volume. The basic obstacle here is that these number fields cannot accommodate metric structures and the notion of dimension. The very difficult task, then, is to reproduce, at scales much larger than the Planck scale, quantum physics as we know it.

In this note we take a first step in connecting Finite Quantum Mechanics FQM [16] to a continuum quantum mechanics of a rather particular type. Indeed we show that FQM is a consistent and *exact* discretization of holomorphic quantum mechanics on toroidal phase spaces for arbitrary moduli, thereby establishing a possible link to rational conformal field theories on the torus. We extend the work of ref. [15] to torus discretizations of any length.

The plan of the paper is as follows: in the next section we review holomorphic quantum mechanics on the (continuum) torus; we then discuss finite quantum mechanics and end by discussing some properties of harmonic oscillator eigenfunctions on these spaces and further perspectives.

2 Holomorphic Quantum Mechanics

We start by describing holomorphic quantum mechanics on the torus [17] (cf. also Leboeuf and Voros in ref. [5]). The torus of complex modulus $\tau \in \mathbb{C}$ is defined as the coset space $\Gamma = \mathbb{C}/\mathbb{L}$, where $\mathbb{L}$ is the integer lattice $\mathbb{L} = \{m_1 + \tau m_2 | (m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}\}$. The torus Γ is the set of points of the complex plane $\mathbb{C}$, $z = q + \tau p,\ q, p, \in [0, 1]$. The symplectic structure of $\mathbb{C}$ induces on Γ the (symplectic) form

$$\Omega = -\frac{1}{2\mathrm{i}} dz \wedge d\overline{z} = \tau_2 dq \wedge dp, \tag{1}$$

where $\tau = \tau_1 + \mathrm{i}\tau_2$. The corresponding group of symplectic transformations is $SL(2, \mathbb{R})$ acting on $(q, p) \bmod 1$ [2]. To define holomorphic quantum mechanics on Γ

we start by the classical evolution in the phase space Γ under elements of $SL(2,\mathbb{R})$. The most general quadratic Hamiltonian

$$\mathcal{H} = \frac{\tau_2}{2}(q\,,\,p)\begin{pmatrix} -c & a \\ a & b \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} \tag{2}$$

leads to the evolution equations

$$\frac{d}{dt}(q\ ,\ p) = (q\ ,\ p)\begin{pmatrix} a & c \\ b & -a \end{pmatrix} \tag{3}$$

which are immediately integrated to

$$(q(t)\ ,\ p(t)) \equiv (q(0)\ ,\ p(0))\mathcal{R}(t) \bmod 1 \tag{4}$$

with $\mathcal{R}(t) \in SL(2,\mathbb{R})$ given by

$$\mathcal{R}(t) = \exp\left[t\begin{pmatrix} a & c \\ b & -a \end{pmatrix}\right] \tag{5}$$

The *quantum mechanical* evolution, with Weyl ordering as in eq. (2), is also simple and leads to

$$(\hat{q}(t)\ ,\ \hat{p}(t)) = (\hat{q}(0)\ ,\ \hat{p}(0))\mathcal{R}(t) \tag{6}$$

The position and momentum operators, $\hat{q}$ and $\hat{p}$ satisfy

$$[\hat{q},\hat{p}] = \frac{\mathrm{i}\hbar}{\tau_2} \tag{7}$$

From (6) and the Heisenberg equations of motion we have that

$$\mathcal{U}(t)(\hat{q}(0)\ ,\ \hat{p}(0))\mathcal{U}^{-1}(t) = (\hat{q}(0)\ ,\ \hat{p}(0))\mathcal{R}(t) \tag{8}$$

with $\mathcal{U}(t)$ the evolution operator

$$\mathcal{U}(t) = \exp\left[\frac{\mathrm{i}t}{\hbar}\frac{\tau_2}{2}(\hat{q}\ ,\ \hat{p})\begin{pmatrix} -c & a \\ a & b \end{pmatrix}\begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix}\right] \tag{9}$$

This last relation, as we shall see, defines a representation of $SL(2,\mathbb{R})$. The Hilbert space of quantum mechanics on the torus Γ consists of functions (to be more precise this is a space of sections of a $U(1)$ bundle over Γ) given by series

$$f(z) = \sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{\mathrm{i}\pi n^2\tau + 2\pi\mathrm{i}nz} \tag{10}$$

with norm [19]

$$||f||^2 = \int \mathrm{e}^{-2\pi y^2/\tau_2}|f(z)|^2 dxdy, \ \ \tau_2 > 0 \tag{11}$$

On this space consider the action of the operators $\mathcal{S}_b$ and $\mathcal{T}_a$

$$\begin{aligned} (\mathcal{S}_b f)(z) &= f(z+b), \ \ \forall f \in \mathbb{H}_\Gamma \\ (\mathcal{T}_a f)(z) &= \mathrm{e}^{\mathrm{i}\pi a^2\tau + 2\pi \mathrm{i}z} f(z+a\tau), \ \ a,b \in \mathbb{R} \end{aligned} \tag{12}$$

which satisfy the fundamental Weyl commutation relations (CR), the integrated form of Heisenberg CR,

$$\mathcal{S}_b \mathcal{T}_a = \mathrm{e}^{2\pi \mathrm{i} ab} \mathcal{T}_a \mathcal{S}_b \tag{13}$$

The operators $\mathcal{S}$ and $\mathcal{T}$ are so chosen that the classical Jacobi theta function [19]

$$\theta(z|\tau) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\pi n^2\tau + 2\pi \mathrm{i} nz} \tag{14}$$

is invariant under $\mathcal{S}_1$ and $\mathcal{T}_1$.

The space $\mathbb{H}_\Gamma$ carries an infinite dimensional, unitary, irreducible representation of the Heisenberg group defined as

$$\mathcal{W}(\lambda, a, b)f = \lambda \mathcal{T}_a \mathcal{S}_b f, \;\; \lambda \in U(1), a, b \in \mathbb{R}, \; \forall f \in \mathbb{H}_\Gamma \tag{15}$$

with composition law

$$\mathcal{W}(\lambda, a, b)\mathcal{W}(\lambda^{'}, a^{'}, b^{'}) = \mathcal{W}(\lambda\lambda^{'}\mathrm{e}^{2\pi \mathrm{i} ba^{'}}, a + a^{'}, b + b^{'}) \tag{16}$$

In holomorphic quantum mechanics on the torus [17], $\hat{q}$ and $\hat{p}$ are given by

$$\hat{q} = -\mathrm{i}\partial_z, \;\; \hat{p} = -2\pi z + \mathrm{i}\tau\partial_z \tag{17}$$

and thus

$$\begin{aligned} \mathcal{S}_1 &= \mathrm{e}^{\mathrm{i}\hat{q}} \\ \mathcal{T}_1 &= \mathrm{e}^{-\mathrm{i}\hat{p}} \end{aligned} \tag{18}$$

where we have chosen $\hbar = 2\pi\tau_2$.

We are ready now to describe the metaplectic representation of $SL(2, \mathbb{R})$ on the space $\mathbb{H}_\Gamma$. For every $(q,\, p) \in \Gamma$ the evolution operator, $\mathcal{U}(t)$, (cf. eq. (9)), satisfies the relation (cf. eqs. (8),(5))

$$\mathcal{U}_{\mathcal{R}}^{-1}(t)\mathcal{J}_{q,p}\mathcal{U}_{\mathcal{R}}(t) = \mathcal{J}_{(q\;\, p)\mathcal{R}(t)} \tag{19}$$

where

$$\mathcal{J}_{q,p} \equiv \mathrm{e}^{\mathrm{i}\left(-q\hat{p} + p\hat{q}\right)} \tag{20}$$

is an element of the Heisenberg group acting on $\mathbb{H}_\Gamma$.

The metaplectic representation [17, 20] of $SL(2, \mathbb{R})$ is defined by eq. (19) and, in general, is a projective representation.

3 Finite Quantum Mechanics

We now recall the basic features of FQM and its relation to the holomorphic QM.

The torus phase space has been the simplest prototype for studying classical and quantum chaos [2, 3, 4, 5]. Discrete elements of $SL(2, \mathbb{R})$, i.e. elements of the

modular group $SL(2,\mathbb{Z})$, are studied on discretizations of the torus with rational coordinates of the same denominator l, $(q,p)=(n_1/l,n_2/l)\in\Gamma$, $n_1,n_2,l\in\mathbb{Z}$ and their periodic trajectories mod 1 are examined studying the periods of elements $\mathcal{A}\in SL(2,\mathbb{Z})$ mod l. The action mod 1 becomes mod l on an equivalent torus, $(n_1,n_2)\in l\Gamma$. The classical motion of such discrete dynamical systems is usually "maximally" disconnected and chaotic [3, 5].

FQM is the quantization of these discrete linear maps and the corresponding one-time-step evolution operators $\mathcal{U}_{\mathcal{A}}$ are $l\times l$ unitary matrices called *quantum maps*. In the literature [4, 5] these maps are determined semi-classically. In ref. [15, 16] the exact quantization of $SL(2,\mathbb{F}_p)$, where $\mathbb{F}_p$ is the simplest finite field of p elements with p a prime number was studied in detail. In the following we shall extend the results of ref. [15] to $l=p^n$ and we shall discuss the case of arbitrary integer l.

Consider the subspace $\mathbb{H}_l(\Gamma)$ of $\mathbb{H}_\Gamma$ with periodic Fourier coefficients $\{c_n\}_{n\in\mathbb{Z}}$ of period l

$$c_n=c_{n+l}\ \ n\in\mathbb{Z},\ \ l\in\mathbb{N} \tag{21}$$

The space $\mathbb{H}_l(\Gamma)$ is l-dimensional and there is a discrete Heisenberg group [18], with generators $\mathcal{S}_{1/l}$ and $\mathcal{T}_1$ acting as [17, 19]

$$\begin{aligned}(\mathcal{S}_{1/l}f)(z) &= \sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{2\pi\mathrm{i}n/l}\mathrm{e}^{2\pi\mathrm{i}nz+\pi\mathrm{i}n^2\tau}\\ (\mathcal{T}_1 f)(z) &= \sum_{n\in\mathbb{Z}} c_{n-1}\mathrm{e}^{2\pi\mathrm{i}nz+\pi\mathrm{i}n^2\tau},\ \ c_n\in\mathbb{C}\end{aligned} \tag{22}$$

On the l-dimensional subspace of vectors $(c_1,\ldots,c_l)$ the two generators are represented by

$$(\mathcal{S}_{1/l})_{n_1,n_2}=Q_{n_1,n_2}=\omega^{(n_1-1)}\delta_{n_1,n_2}$$

$$(\mathcal{T}_1)_{n_1,n_2}=P_{n_1,n_2}=\delta_{n_1-1,n_2} \tag{23}$$

with $\omega=\exp(2\pi\mathrm{i}/l)$. The Weyl relation becomes

$$QP=\omega PQ \tag{24}$$

and the Heisenberg group elements are

$$\mathcal{J}_{r,s}=\omega^{r\cdot s/2}P^rQ^s \tag{25}$$

In the literature the metaplectic representation of $SL(2,l)$, (the group of 2×2, integer valued matrices mod l), is known for $l=p^n$ [21][‖]

The Weyl-Fourier form of $\mathcal{U}_{\mathcal{A}}$ is [16]

$$\mathcal{U}_{\mathcal{A}}=\frac{\sigma(1)\sigma(\delta)}{p}\sum_{r,s=0}^{p-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{p}[br^2+(d-a)rs-cs^2]/2\delta}\mathcal{J}_{r,s} \tag{26}$$

[‖] The representation theory of the symplectic group $SL(2,\mathbb{F}_{p^n})$ may be found in ref. [22].

where

$$\mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{F}_p), \;\; \delta = 2 - a - d$$

$$\sigma(a) = \frac{1}{\sqrt{p}} \sum_{r=0}^{p-1} \omega^{ar^2} = (a|p)\mathfrak{p} \tag{27}$$

$(a|p)$ is the Jacobi symbol [1] and

$$\mathfrak{p} = \begin{cases} 1 & p = 4k+1 \\ \mathrm{i} & p = 4k-1 \end{cases}$$

All the operations in the exponent are carried out in the field $\mathbb{F}_p$. If $\delta \equiv 0 \bmod p$ we use the trick

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \tag{28}$$

and the fact that $\mathcal{U}_\mathcal{A}$ is a representation (cf. ref. [16]).

In ref. [15] the eigenproblem for the generators of the "rotation subgroup" of $SL(2,\mathbb{F}_p)$, $O_2(p)$, was solved and an explicit list of the generators $\mathcal{R}_0$ for primes $p < 20\,000$ was given. The spectrum of $\mathcal{R}_0$ is linear and all the eigenvectors, which are real, were found analytically for primes $p = 4k+1$. In fact they are appropriately weighted Hermite polynomials over the finite field $\mathbb{F}_p$ [15, 23]. All of them turned out to be extended, in the sense that their support is the full set $\mathbb{F}_p$ and their components randomly distributed.

The first step to generalize the results of ref. [15] for $O_l(2)$ is to consider integers $l = p^n$, powers of primes. We shall need the explicit form of $\mathcal{U}_\mathcal{A}$ for $l = p$ because this is immediately generalized to $l = p^n$

$$(\mathcal{U}_\mathcal{A})_{n_1,n_2} = \frac{1}{\sqrt{p}} (-2c|p)\mathfrak{p}\omega^{-[a(n_1-1)^2 + d(n_2-1)^2 - 2(n_1-1)(n_2-1)]/2c} \tag{29}$$

for $c \not\equiv 0 \bmod p$ (otherwise apply eq. (28)).

Imposing eq. (26) for $l = p^n$ we need the Gauß sum**

$$\mathfrak{G}(k,l) = \frac{1}{\sqrt{l}} \sum_{r=0}^{l-1} \mathrm{e}^{2\pi \mathrm{i} k r^2/l} \tag{30}$$

It enjoys the property

$$\mathfrak{G}(k,p^n) = p\mathfrak{G}(k,p^{n-2}) \tag{31}$$

which implies that

$$\mathfrak{G}(k,p^{2m}) = p^m \tag{32}$$

**A complete study of this sum for arbitrary integer l can be found in the chapter "Cyclotomic Fields" of S. Lang in ref. [1]; cf. also ref. [24].

so, for $n = 2m$,

$$(\mathcal{U}_{\mathcal{A}})_{n_1,n_2} = \frac{1}{\sqrt{p^{2m}}} \exp\left[-\left(\frac{2\pi \mathrm{i}}{p^{2m}}\left[a(n_1-1)^2 + d(n_2-1)^2 - 2(n_1-1)(n_2-1)\right]/2c\right)\right] \tag{33}$$

with $c \not\equiv 0 \bmod p$. For odd powers of p, $n = 2m+1$,

$$\mathfrak{G}(k, p^{2m+1}) = p^m \mathfrak{G}(k, p) \tag{34}$$

so we have only to replace p by p^{2m+1} in eq. (29) and $1/2c$ is taken mod p^{2m+1}.

The above results can be deduced also from the work of S. Tanaka on the representations of $SL(2, p^n)$ [21].

For practical calculations of spectra and eigenvectors of $O_{p^n}(2)$ for various primes one has to determine the corresponding generators $\mathcal{R}_0$. Here we explicitly present their construction. In ref. [15] $\mathcal{R}_0$ was found in the case of $p = 4k+1$ once a primitive element of $\mathbb{F}_p$, $\mathfrak{g}$, is given.

$$\mathcal{R}_0 = \begin{pmatrix} \dfrac{\mathfrak{g}+\mathfrak{g}^{-1}}{2} & \dfrac{\mathfrak{g}^{-1}-\mathfrak{g}}{2\mathfrak{t}} \\ \dfrac{\mathfrak{g}-\mathfrak{g}^{-1}}{2\mathfrak{t}} & \dfrac{\mathfrak{g}+\mathfrak{g}^{-1}}{2} \end{pmatrix} \tag{35}$$

here $\mathfrak{t} \equiv \mathfrak{g}^k \bmod p$, $\mathfrak{t}^2 \equiv -1 \bmod p$ and all operations in the entries of eq. (35) are performed mod p.

The set of integers mod p^n does not form a finite field, but there is a multiplicative subgroup, composed of all the integers, $\not\equiv 0 \bmod p$. A known theorem states that, if $\mathfrak{g}^{p-1} \not\equiv 1 \bmod p^n$, then $\mathfrak{g}$ is a generator of this cyclic group with order $\phi(p^n) = p^n - p^{n-1}$. If $p = 4k+1$, $\phi(p^n)$ is divisible by 4 and there is an element $\mathfrak{t}$ ($\mathfrak{t}^2 \equiv -1 \bmod p^n$), $\mathfrak{t} \equiv \mathfrak{g}^{\phi(p^n)/4}$. In this case $\mathcal{R}_0$ is given by eq. (35) where all the operations are mod p^n.

In the case $p = 4k-1$ we need to know a primitive element $\mathsf{w} = \mathsf{w}_1 + \mathrm{i}\mathsf{w}_2$ of $\mathbb{F}_{p^2}$ [16, 15]. The corresponding generator of $O_p(2)$ is

$$\mathcal{R}_0 = \begin{pmatrix} u_1 & u_2 \\ -u_2 & u_1 \end{pmatrix}$$

$$u_1 + \mathrm{i}u_2 = \frac{\mathsf{w}^2}{\mathfrak{g}}, \quad \mathfrak{g} = \mathsf{w}\overline{\mathsf{w}} \in \mathbb{F}_p \tag{36}$$

here $\overline{\mathsf{w}} = \mathsf{w}_1 - \mathrm{i}\mathsf{w}_2 \equiv \mathsf{w}^p \bmod p$ and $\mathfrak{g}$ can be shown to be a primitive element of $\mathbb{F}_p$. A list of $\mathcal{R}_0$ and w for all primes $p = 4k-1 < 20\,000$ can be found in ref. [15].

For $l = p^n$, $p = 4k-1$, we can find primitive elements $\mathsf{w} \in \mathbb{F}_{p^2}$ such that $\mathfrak{g} = \mathsf{w}\overline{\mathsf{w}}$ has the property $\mathfrak{g}^{p-1} \not\equiv 1 \bmod p$ and the corresponding generator $\mathcal{R}_0$ is given by eq. (36) where all the operations are performed mod p^n.

From the above one can find that, for $l = p^n = (4k+1)^n$ the period of the generator is $\phi(p^n) = p^n - p^{n-1}$, while, for $l = p^n = (4k-1)^n$, the period is $p^n + p^{n-1}$.

For arbitrary $l = \prod_{i=1}^{s} p_i^{n_i}$ $SL(2, l) = \otimes_{i=1}^{s} SL(2, p_i^{n_i})$ [1, 7], It is known that $SL(2, l)$ is the coset space $SL(2, \mathbb{Z})/\Gamma_l$, where Γ_l is the set of matrices $\mathcal{A} \in SL(2, \mathbb{Z})$,

such that $\mathcal{A} = \pm I$ mod l. This is a normal subgroup of $SL(2,\mathbb{Z})$ and is called the *principal congruent subgroup mod l.* It plays an important role in the geometry of Riemann surfaces and the classification of modular forms (cf. the article by D. Zagier in ref. [10]). $SL(2,l)$ consists of *nested* sequences, in the sense that $SL(2,l) \subset SL(2,l^{'})$ when $l^{'} \equiv 0$ mod l.

The metaplectic representation, eq. (26), can be extended to any l, once $\delta = 2 - a - d \not\equiv 0$ mod p_i (for any p_i); the Gauß sums can be easily evaluated (cf. S. Lang in ref. [1])–unfortunately, there isn't any simple, *unique* answer for arbitrary $\mathcal{A} \in SL(2,l)$. For the class of $\mathcal{A}$'s, of the form

$$\mathcal{A} = \left(\begin{array}{cc} even & odd \\ odd & even \end{array} \right) \quad \text{or} \quad \mathcal{A} = \left(\begin{array}{cc} odd & even \\ even & odd \end{array} \right) \tag{37}$$

Hannay and Berry [4] have written down the semiclassical form of $\mathcal{U}_{\mathcal{A}}$. It is not difficult to see that the metaplectic representation, eq. (26),leads to the same results. For the other forms of $\mathcal{A}$ the answer for $\mathcal{U}_{\mathcal{A}}$ does not have the same form for all l. Our main interest is the harmonic oscillator subgroup, $O_l(2) \subset SL(2,l)$. As we mentioned before, $SL(2,l)$ can be decomposed into a tensor product of $SL(2,p_i^{n_i})$, $i = 1,\ldots,s$ over the prime factors of l. The same happens for $O_l(2)$, which is an abelian group, with s cycles and with generators $\mathcal{R}_0(p_i^{n_i})$. Its representations may thus be obtained by tensoring powers of $\mathcal{U}_{\mathcal{R}_0(p_i^{n_i})}$.

4 Perspectives

We discuss finally the construction of the eigenstates of the harmonic oscillator subgroup (mod l). These are presumably the building blocks of field theories (and string theories) on discretized toroidal phase spaces. It is enough to determine the eigenstates (and eigenvalues) of $O_{p^n}(2)$ for any prime p and (positive) integer n. From the construction of $\mathcal{R}_0$ and their diagonalized form

$$\left(\begin{array}{cc} a & b \\ -b & a \end{array} \right) = \mathcal{L} \left(\begin{array}{cc} a - \mathfrak{t}b & 0 \\ 0 & a + \mathfrak{t}b \end{array} \right) \mathcal{L}^{-1} \quad a^2 + b^2 \equiv 1 \bmod p^n \tag{38}$$

where

$$\mathcal{L} = \frac{1}{2\mathfrak{t}} \left(\begin{array}{cc} 1 & 1 \\ -\mathfrak{t} & \mathfrak{t} \end{array} \right) \tag{39}$$

with $\mathfrak{t}^2 \equiv -1 \bmod p^n$, $\sqrt{2\mathfrak{t}} \equiv (1+\mathfrak{t}) \bmod p^n$, we diagonalize the corresponding $\mathcal{U}_\Delta$, where

$$\Delta = \left(\begin{array}{cc} a - \mathfrak{t}b & 0 \\ 0 & a + \mathfrak{t}b \end{array} \right)$$

As was shown in ref. [15], $\mathcal{U}_\Delta$ is a *circulant* matrix for $l = p$ (and the same happens here as well, in each sector p^n) for $p = 4k+1$: its first row is $e_1 = (1,0,\ldots,0)$ and each subsequent row has the element 1 shifted by $\mathfrak{g}^{-1}$ mod p^n positions to the right from the last. The eigenvectors of $\mathcal{U}_\Delta$ are easily found to be the multiplicative characters of the set of integers mod p^n, extending the results of ref. [15]. The

eigenvalues of $\mathcal{U}_\Delta$ are roots of unity, of order $p^n - p^{n-1}$ and p^{n-1} of them must be degenerate.

For $p = 4k - 1$, it is possible to find *directly* the corresponding eigenvectors of $\mathcal{U}_\Delta$. They are the multiplicative characters of the rotation group $O_{p^n}(2)$, while the eigenvalues are roots of unity of order $p^n + p^{n-1}$.

We close this note by writing $\mathcal{U}_\mathcal{A}$, $\mathcal{A} \in SL(2, l)$, in terms of holomorphic operators on $\mathbb{H}_l(\Gamma)$. Define the elements of the Heisenberg group

$$\mathcal{J}_{r,s} = \exp\left(\mathrm{i}\left[-r\hat{p} + s\hat{q}\right]\right),\ \ r, s = 1, \ldots, l \tag{40}$$

with

$$\begin{aligned} \hat{p} &= -2\pi z + \mathrm{i}\tau\partial_z \\ \hat{q} &= \frac{-\mathrm{i}}{l}\partial_z \\ [\hat{q}, \hat{p}] &= \frac{2\pi\mathrm{i}}{l} \end{aligned} \tag{41}$$

In eq. (26) we substitute P and Q with $\mathcal{T}_1$ and $\mathcal{S}_{1/l}$ respectively and carry out first the summation over s and then over r. We end up with

$$\mathcal{U}_\mathcal{A} = \exp\left(-\frac{2\pi\mathrm{i}}{l}\frac{\delta}{2b}\left(\frac{l}{2\pi}\hat{p}\right)^2\right)\exp\left(-\frac{2\pi\mathrm{i}}{l}\frac{1}{2b}\left[(1-a)\frac{l}{2\pi}\hat{p} + b\frac{l}{2\pi}\hat{q}\right]^2\right) \tag{42}$$

where the operators in the exponents have integer eigenvalues. We assume $\delta, b \not\equiv 0 \mod p$.

Finally we address the issue of localization of the eigenstates of $\mathcal{U}_{\mathcal{R}_0}$ for the harmonic oscillator. For $l = p^n$ this operator is represented by a $p^n \times p^n$ unitary matrix of period $p^n \mp p^{n-1}$ for $p = 4k \pm 1$.

Higher powers of $\mathcal{R}_0$ (higher degeneracy but *smaller* period) have classical orbits that are localized in phase space (intuitively understandable: since the period is smaller the orbits wander less in phase space)–and the quantum eigenstates follow suit. A nice example is provided by the finite Fourier transform; set $\mathfrak{F} = \mathcal{R}_0^{\phi(p^n)/4}$, with $\mathfrak{F}^4 = I$. The quantum map $\mathcal{U}_\mathfrak{F}$, the finite Fourier transform, is known to possess localized eigenstates [25]

$$\varphi_k(j) = \left(\frac{\partial}{\partial x}\right)^k \left[e^x\theta\left(\frac{j}{l} - x\sqrt{\frac{2}{\pi l}}\middle|\tau = \frac{\mathrm{i}}{l}\right)\right]\Bigg|_{x=0} \qquad j = 0, 1, \ldots, l-1;\ k = 0, 1, \ldots, \tag{43}$$

These states are not orthogonal and, surprisingly, are discrete approximations of the continuum harmonic oscillator states.

The ground state, $\varphi_0(j)$, is a Gaussian and the action of $\mathcal{U}_{\mathcal{R}_0}$ on it is maximally dispersive. However, since $\mathcal{U}_{\mathcal{R}_0}$ has a finite period, the evolution of the ground state is periodic.

We end with some open problems. The above findings suggest that the naive continuum (*not* classical) limit of the eigenstates of $\mathcal{U}_{\mathcal{R}_0}$ doesn't lead to sensible

results for integer sequences, $l_n = p^n$, for a *fixed* prime p and $n = 1, \ldots$. For most of the extended states, this limit exists in the Hilbert space $\mathbb{H}_\Gamma$ and is zero, since $\sum_{m=1}^{l} |c_m|^2 = 1$ and $c_m \approx O(1/\sqrt{l})$; it may be possible to find suitable sequences, $\mathcal{U}_{\mathcal{R}_0^{r_n}}$, such that only some localized states survive in the limit. On the other hand, from the construction of the $p-$adic numbers, $\mathbb{Q}_p$ and $SL(2, \mathbb{Q}_p)$ [9], it is known that there does exist another "continuum" limit, in the $p-$adic numbers, which is called *projective* and is related to $p-$adic quantum mechanics for p^n, $n \to \infty$ (cf. Y. Meurice in ref. [9] and references therein). However the relation between the finite fields and the $p-$adic numbers is far from obvious and the relation between our construction and that valid for the $p-$adics not known at the moment.

For higher dimensional phase spaces the construction of the metaplectic representations of the symplectic group, $Sp(2D, l)$ (where D is the dimension of the space), follows similar lines.

Regarding "practical" applications, the eigenstates of $\mathcal{U}_{\mathcal{R}_0}$ can be used to construct finite, orthogonal sets of wavelets over finite fields [15], appropriate for analyzing *local* time-frequency or position-scale statistics of images. Another area is coding theory (especially cryptography). Some standard codes are linear or polynomial transformations over finite fields [12]. Our present work could be useful in "quantizing" linear codes or writing codes executable by quantum computers [26] as well as implementing algorithms for specifically quantum computation [27].

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